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On a problem about quadrant-depth ☆

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ABSTRACT

Let P be a set of n points in general position in the plane. For every $x \in P$ let $D(x, P)$ be the maximum number such that there are at least $D(x, P)$ points of P in each of two opposite quadrants determined by some two perpendicular lines through x. Define $D(P) = \max_{x \in P} D(x, P)$. In this paper we show that $D(P) \geqslant c|P|$ for every set P in general position in the plane where c is an absolute constant that is strictly greater than $\frac{1}{8}$. This answers a question raised by Stefan Felsner, and, as it turns out, also independently raised by Brönnimann, Lenchner, and Pach.

1. Introduction

In [1], Brönnimann, Lenchner, and Pach define the notion of *opposite-quadrant* depth of a point in a point set. Specifically, let P be a set of n points in general position in the plane. The *opposite-quadrant* depth of a point x in P is the maximum number $opp(x, P)$ such that the horizontal and vertical lines through x define two opposite closed quadrants each containing at least $opp(x, P)$ points of P. The main result established in [1] is that any set P of n points in general position in the plane has a point $x \in P$ such that $opp(x, P) \geqslant \frac{1}{8}|P|$ and that this bound is best possible.

The following theorem from [2] generalizes the main result in [1]:

Theorem 1. *(See [2].) Let P be a set of n points in general position in the plane. Then there exists a point $p \in P$ such that the four (closed) quadrants Q_1, Q_2, Q_3, and Q_4, determined by the horizontal and vertical lines through p (where Q_1 and Q_3 are opposite) satisfy:*

$$\min(|Q_1 \cap P|, |Q_3 \cap P|) + \min(|Q_2 \cap P|, |Q_4 \cap P|) \geqslant \frac{1}{4}|P|.$$

For more on the (rather short) history, background and related problems we refer the reader to [1]. In an attempt to find an interesting variant of the opposite-quadrant problem, one can consider the following definition. Let P be a set of n points in general position in the plane. For $x \in P$ we define $D(x, P)$ to be the maximum number such that there are at least $D(x, P)$ points of P in each of two opposite quadrants determined by *some* two perpendicular lines through x. Let $D(P) = \max_{x \in P} D(x, P)$. We note that in this paper we will not specify whether the quadrants are closed or open as we will ignore additive factors of $O(1)$ in our estimates.

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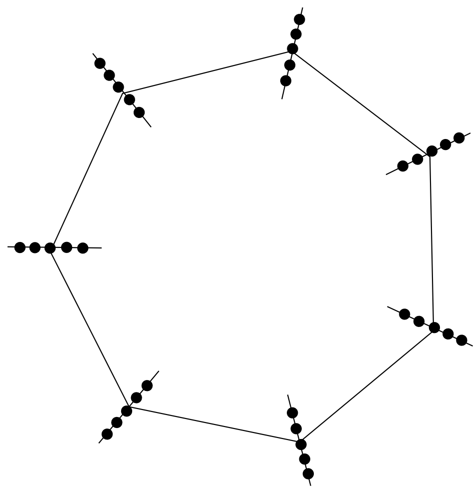

Fig. 1. A set P with $D(x, P) \leqslant n/7$ for every $x \in P$.

Clearly, for any set P and a point $x \in P$, we have $opp(x, P) \leqslant D(x, P)$. Therefore, by Theorem 1, $D(P) \geqslant \frac{1}{8}|P|$ for every set P of points in general position in the plane. The following problem was raised by Stefan Felsner [3] and independently also by Brönnimann, Lenchner, and Pach:

Do there exist sets P such that $D(P) = \frac{1}{8}|P|$, or does there exist a constant $c > \frac{1}{8}$ such that $D(P) \geqslant c|P|$ for every set P?[1]

We note already that despite the similarity to classical partition theorems in the plane and higher dimensions by perpendicular hyper-planes (see for example [4] and [5]), the condition on the perpendicular lines to pass through a point of the set makes the problem completely different and in a way more difficult.

The following simple construction shows that there are sets P for which $D(x, P) \leqslant \frac{1}{7}n$ for every $x \in P$. Let Q be a regular 7-gon centered at the origin. Very close to each vertex v of Q position $n/7$ points along a line through the origin and v. In order for the resulting set of points to be in general position, slightly perturb the points just to avoid unnecessary collinearities. It is left to the reader to verify that the resulting set P of n points satisfies $D(x, P) \leqslant n/7$ for every $x \in P$. See Fig. 1 and imagine the points corresponding to each vertex of Q much closer to each other and to the respective vertex of Q.

The main result in this paper is the existence of a constant c strictly greater than $\frac{1}{8}$ such that $D(P) \geqslant c|P|$ for every set P.

Theorem 2. *For every finite set P of points in general position in the plane we have $D(P) \geqslant (\frac{1}{8} + \frac{1}{8 \cdot 39})|P|$.*

2. Proof of Theorem 2

In what follows P will denote a set of n points in the plane. For a directed line ℓ we denote by $H_L(\ell)$ the open half-plane bounded by ℓ which lies to the left of ℓ. Similarly, $H_R(\ell)$ denotes the open half-plane bounded by ℓ which lies to the right of ℓ.

Let ℓ_1 and ℓ_2 be two directed lines that meet at a single point. The *front wedge* determined by ℓ_1 and ℓ_2 is the set $H_R(\ell_1) \cap H_L(\ell_2)$. We denote this region by $W_F(\ell_1, \ell_2)$. Similarly, the *back wedge* determined by ℓ_1 and ℓ_2 is the set $H_L(\ell_1) \cap H_R(\ell_2)$ which we denote by $W_B(\ell_1, \ell_2)$. In a similar manner we define $W_L(\ell_1, \ell_2) = H_L(\ell_1) \cap H_L(\ell_2)$ and $W_R(\ell_1, \ell_2) = H_R(\ell_1) \cap H_R(\ell_2)$, the *left wedge* and the *right wedge*, respectively, determined by ℓ_1 and ℓ_2 (see Fig. 2).

Lemma 1. *Let P be a set of n points in general position in the plane. Let ℓ_1 and ℓ_2 be two perpendicular directed lines and assume that $|H_R(\ell_1) \cap P| \leqslant \frac{1}{4}n$ and $|H_L(\ell_2) \cap P| \leqslant \frac{1}{4}n$. Then there exists a point $x \in P$ such that $D(x, P) \geqslant \frac{1}{8}n + \frac{1}{8}|P \cap W_F(\ell_1, \ell_2)|$.*

Proof. Let $t = |P \cap W_F(\ell_1, \ell_2)|$. Rotate the plane so that ℓ_1 is horizontal and points in the direction of the negative part of the x-axis, while ℓ_2 is vertical and points in the direction of the positive part of the y-axis.

Let P' be the subset of P which consists of the (typically, not disjoint) union of the leftmost $\frac{1}{4}n$ points of P, the rightmost $\frac{1}{4}n$ points of P, the topmost $\frac{1}{4}n$ points of P, and the lower $\frac{1}{4}n$ points of P.

Because of the condition on ℓ_1 and ℓ_2, we know that there are t points of P, namely $P \cap W_F(\ell_1, \ell_2)$, that lie in the intersection of the leftmost $\frac{1}{4}n$ points of P and the topmost $\frac{1}{4}n$ points of P. It follows that $P'' = P \setminus P'$ consists of at least t points. See Fig. 3.

[1] In this paper we will ignore questions of divisibility of n whenever they are relevant, and thus all equalities and inequalities involving n or $|P|$ are true up to an additive factor of $O(1)$ which will in fact be at most 1 in absolute value.

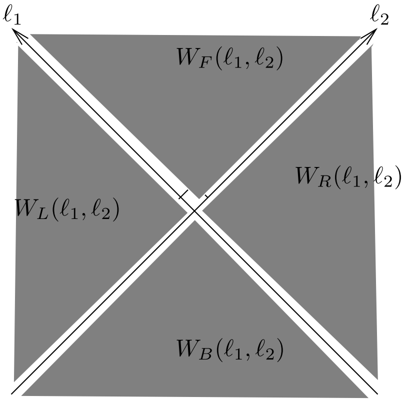

Fig. 2. α-double-wedge $W(\ell_1, \ell_2)$.

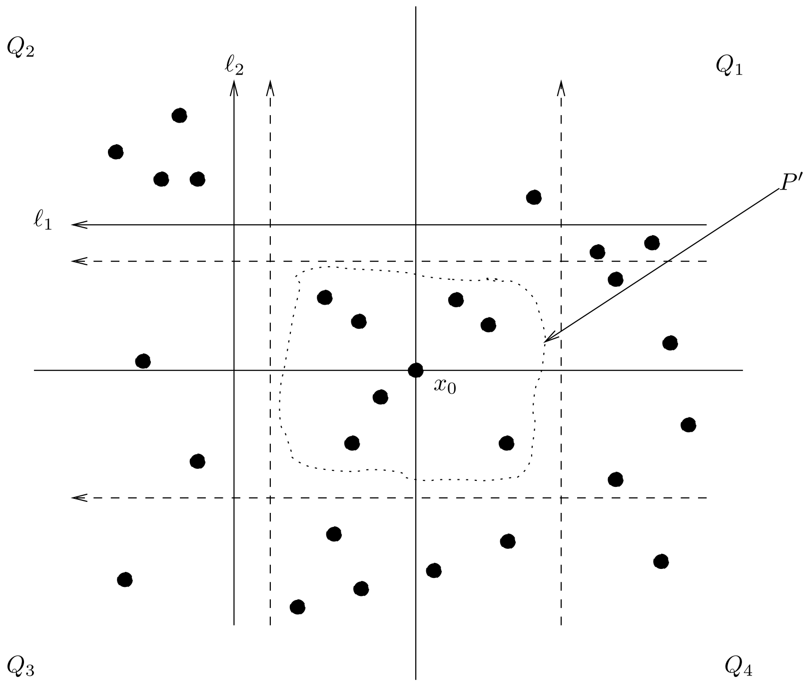

Fig. 3. Lemma 1.

Let x be any point in P'' and let Q_1, Q_2, Q_3, and Q_4, be the four quadrants determined by the vertical and horizontal lines through x. It is not hard to see, although may require a minute of thought, that

$$\min(|Q_1 \cap P'|, |Q_3 \cap P'|) + \min(|Q_2 \cap P'|, |Q_4 \cap P'|) \geqslant \frac{1}{4}n.$$

This is essentially the very short argument proving Theorem 1 in [2] and we refer the reader to there.

By Theorem 1, there exists a point $x_0 \in P''$ such that the four quadrants Q_1, Q_2, Q_3, and Q_4, determined by the vertical and horizontal lines through x_0, satisfy

$$\min(|Q_1 \cap P''|, |Q_3 \cap P''|) + \min(|Q_2 \cap P''|, |Q_4 \cap P''|) \geqslant \frac{1}{4}|P''| + O(1) \geqslant \frac{1}{4}t + O(1).$$

We deduce the following:

$$\begin{aligned}&\min(|Q_1 \cap P|, |Q_3 \cap P|) + \min(|Q_2 \cap P|, |Q_4 \cap P|)\\ &\quad = \min(|Q_1 \cap P'| + |Q_1 \cap P''|, |Q_3 \cap P'| + |Q_3 \cap P''|) + \min(|Q_2 \cap P'| + |Q_2 \cap P''|, |Q_4 \cap P'| + |Q_4 \cap P''|)\end{aligned}$$

$$\begin{aligned}
&\geqslant \min\bigl(\lvert Q_1 \cap P'\rvert, \lvert Q_3 \cap P'\rvert\bigr) + \min\bigl(\lvert Q_1 \cap P''\rvert, \lvert Q_3 \cap P''\rvert\bigr) \\
&\quad + \min\bigl(\lvert Q_2 \cap P'\rvert, \lvert Q_4 \cap P'\rvert\bigr) + \min\bigl(\lvert Q_2 \cap P''\rvert, \lvert Q_4 \cap P''\rvert\bigr) \\
&\geqslant \frac{1}{4}n + \frac{1}{4}t + O(1).
\end{aligned}$$

It follows that either each of Q_1 and Q_3 contains $\frac{1}{8}n + \frac{1}{8}t + O(1)$ points of P, or each of Q_2 and Q_4 contains $\frac{1}{8}n + \frac{1}{8}t + O(1)$ points of P, as desired. $\square$

Corollary 1. *Let P be a set of n points in general position in the plane. Let ℓ_1 and ℓ_2 be two perpendicular directed lines with $|H_R(\ell_1) \cap P| = a$, $|H_L(\ell_2) \cap P| = b$, and $|W_F(\ell_1, \ell_2) \cap P| = c$. Then there exists a point $x \in P$ such that $D(x, P) \geqslant \frac{1}{8}n + \frac{1}{8}(c - \max(a - \frac{1}{4}n, 0) - \max(b - \frac{1}{4}n, 0))$.*

Proof. Let ℓ_1' be a line that is parallel to ℓ_1 such that $|H_R(\ell_1') \cap P| = \frac{1}{4}n$. Let ℓ_2' be a line that is parallel to ℓ_2 such that $|H_L(\ell_2') \cap P| = \frac{1}{4}n$. We have $|P \cap W_F(\ell_1', \ell_2')| \geqslant c - \max(a - \frac{1}{4}n, 0) - \max(b - \frac{1}{4}n, 0)$. The result now follows by applying Lemma 1 to the lines ℓ_1' and ℓ_2'. $\square$

Returning to the proof of Theorem 2, let P be a set of n points in general position in the plane and assume that for every $x \in P$ we have $D(x, P) < (\frac{1}{8} + \epsilon)n$, where $\epsilon > 0$ will be determined later to obtain a contradiction. We will choose ϵ that is not greater than $\frac{1}{8\cdot 39}$.

Fix x in P and consider all pairs of perpendicular directed lines ℓ_1 and ℓ_2 through x. It follows easily by a continuity argument that there are two perpendicular directed lines $\ell_1(x)$ and $\ell_2(x)$ through x such that

$$\bigl|P \cap W_L\bigl(\ell_1(x), \ell_2(x)\bigr)\bigr| = \bigl|P \cap W_R\bigl(\ell_1(x), \ell_2(x)\bigr)\bigr|.$$

Let t denote the cardinality $t = |P \cap W_L(\ell_1(x), \ell_2(x))|$. Because $D(x, P) < (\frac{1}{8} + \epsilon)n$, we have $t < (\frac{1}{8} + \epsilon)n$. Moreover, at least one of $|P \cap W_F(\ell_1(x), \ell_2(x))|$ and $|P \cap W_B(\ell_1(x), \ell_2(x))|$ must be smaller than $(\frac{1}{8} + \epsilon)n$. Without loss of generality assume that $s = |P \cap W_B(\ell_1(x), \ell_2(x))| < (\frac{1}{8} + \epsilon)n$.

By Corollary 1, there exists a point $y \in P$ with $D(y, P) \geqslant \frac{1}{8}n + \frac{1}{8}(s - 2\max(s - \frac{1}{8}n + \epsilon n, 0))$. Therefore, as $D(y, P) < (\frac{1}{8} + \epsilon)n$, we deduce that

$$\frac{1}{8}\left(s - 2\max\left(s - \frac{1}{8}n + \epsilon n, 0\right)\right) < \epsilon n. \tag{1}$$

If $s > \frac{1}{8}n - \epsilon n$, then we get $\frac{1}{8}(s - 2(s - \frac{1}{8}n + \epsilon n)) < \epsilon n$. This implies that $s > \frac{1}{4}n - 10\epsilon n$. On the other hand we know that $s < \frac{1}{8}n + \epsilon n$. Hence, $\frac{1}{8} < 11\epsilon$ which contradicts for the choice of ϵ.

Therefore, we necessarily have $s < \frac{1}{8}n - \epsilon n$. From (1) we deduce now that $s < 8\epsilon n$. This implies that $|P \cap W_F(\ell_1(x), \ell_2(x))| > (\frac{3}{4} - 10\epsilon)n$.

We summarize our findings in the next claim:

Claim 1. *For every $x \in P$ we have*:

- *$W_F(\ell_1(x), \ell_2(x))$ contains more than $(\frac{3}{4} - 10\epsilon)n$ points of P.*
- *$W_B(\ell_1(x), \ell_2(x))$ contains less than $8\epsilon n$ points of P.*
- *each of $W_L(\ell_1(x), \ell_2(x))$ and $W_R(\ell_1(x), \ell_2(x))$ contains less than $\frac{1}{8}n - \epsilon n$ points of P.*

From the lower bound on the number of points of P in $W_F(\ell_1(x), \ell_2(x))$, and from the fact that $(\frac{3}{4} - 10\epsilon)n > \frac{1}{2}n$, we can deduce the following very simple yet important observation:

Observation 1. *For every $x, y \in P$ the interior of $W_F(\ell_1(x), \ell_2(x))$ must intersect with the interior of $W_F(\ell_1(y), \ell_2(y))$.*

Let $P_1 \subset P$ be the set of all points $x \in P$ such that the vertical line through x cuts through $W_F(\ell_1(x), \ell_2(x))$ (and therefore also through $W_B(\ell_1(x), \ell_2(x))$). The complementary set $P_2 = P \setminus P_1$ consists of all those points $x \in P$ such that the horizontal line through x cuts through $W_F(\ell_1(x), \ell_2(x))$ (and therefore also through $W_B(\ell_1(x), \ell_2(x))$). By suitably rotating the plane we may assume, by continuity arguments, that $|P_1| = |P_2| = \frac{1}{2}n$. Indeed, consider the lines $\ell_1(x)$ and $\ell_2(x)$ fixed for every $x \in P$, and note that we may assume that each of these lines has a unique direction. Start rotating the plane and observe that the cardinality of P_1 changes by at most one unit at a time. Therefore, we can reach a position where $|P_1| = \frac{1}{2}n$.

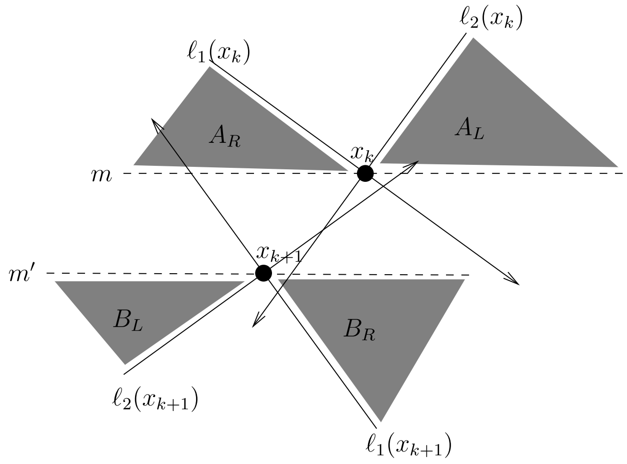

Fig. 4. If $x_{k+1} \in W_R(\ell_1(x_k), \ell_2(x_k))$, then $B_L \cap W_F(\ell_1(x_k), \ell_2(x_k)) = \emptyset$.

We denote the points of P_1 by $x_1, \ldots, x_{\frac{n}{2}}$ according to the decreasing order of their y-coordinates. For every point $x \in P_1$ $W_F(\ell_1(x), \ell_2(x))$ is contained either in the half-plane below the horizontal line through x, or in the half-plane above the horizontal line through x. In the former case we say that x *points downwards* and in the latter case we say that x *points upwards*.

Observe that x_1 must point downwards. This is because if x_1 points upwards, it is impossible for $W_F(\ell_1(x_1), \ell_2(x_1))$ to contain $(\frac{3}{4} - 10\epsilon)n$ points of P (thus contradicting Claim 1), as it does not contain any point of P_1. Here we use the assumption that $10\epsilon < \frac{1}{4}$, and so $(\frac{3}{4} - 10\epsilon)n > \frac{1}{2}n$. Similarly, $x_{\frac{n}{2}}$ must point upwards. Therefore, there exists an index $1 \leqslant k < \frac{n}{2}$, such that x_k points downwards but x_{k+1} points upwards. Because x_k points downwards, $W_F(\ell_1(x_k), \ell_2(x_k))$ cannot contain any of the points $x_1, \ldots, x_k$. Therefore, $W_F(\ell_1(x_k), \ell_2(x_k))$ contains at most $n - k$ points of P. On the other hand, we know that $W_F(\ell_1(x_k), \ell_2(x_k))$ contains more than $\frac{3}{4}n - 10\epsilon n$ points of P. Hence, $k < (\frac{1}{4} + 10\epsilon)n$. Similarly, x_{k+1} points upwards, and therefore $W_F(\ell_1(x_{k+1}), \ell_2(x_{k+1}))$ contains at most $n - (\frac{n}{2} - k) = \frac{n}{2} + k$ points of P. Hence, $k \geqslant (\frac{1}{4} - 10\epsilon)n$.

Let m be the horizontal line through x_k. We know from Claim 1 that $|P \cap W_B(\ell_1(x_k), \ell_2(x_k))| \leqslant 8\epsilon n$. On the other hand, the half-plane bounded above m contains k points of P_1. It follows that there are at least $k - 8\epsilon n \geqslant (\frac{1}{4} - 18\epsilon)n$ points of P_1 above m but outside of $W_B(\ell_1(x_k), \ell_2(x_k))$. Note that the area above m but outside of $W_B(\ell_1(x_k), \ell_2(x_k))$ is included in the union of $W_L(\ell_1(x_k), \ell_2(x_k))$ and $W_R(\ell_1(x_k), \ell_2(x_k))$, and each of $W_L(\ell_1(x_k), \ell_2(x_k))$ and $W_R(\ell_1(x_k), \ell_2(x_k))$ contains less than $(\frac{1}{8} + \epsilon)n$ points of P. Denote by A_L the area above m that is contained in $W_L(\ell_1(x_k), \ell_2(x_k))$. Denote by A_R the area above m that is contained in $W_R(\ell_1(x_k), \ell_2(x_k))$. Therefore, there are more than $(\frac{1}{8} - 19\epsilon)n$ points of P_1 in each of A_L and A_R.

In a similar way one defines m' to be the horizontal line through x_{k+1} and let B_L denote the area below m' that is contained in $W_L(\ell_1(x_{k+1}), \ell_2(x_{k+1}))$, and let B_R denote the area below m' that is contained in $W_R(\ell_1(x_{k+1}), \ell_2(x_{k+1}))$. Then one shows that there are more than $(\frac{1}{8} - 19\epsilon)n$ points of P_1 in each of B_L and B_R. See Fig. 4.

Claim 2. *x_{k+1} must be contained in the open interior of $W_F(\ell_1(x_k), \ell_2(x_k))$.*

Proof. Assume to the contrary that x_{k+1} is not contained in the interior of $W_F(\ell_1(x_k), \ell_2(x_k))$. Then without loss of generality x_{k+1} is contained in $W_R(\ell_1(x_k), \ell_2(x_k))$. By Observation 1, $W_F(\ell_1(x_k), \ell_2(x_k))$ intersects with the interior of $W_F(\ell_1(x_{k+1}), \ell_2(x_{k+1}))$. Therefore, $\ell_2(x_{k+1})$ must lie below x_k (see Fig. 4). Now it is easy to see the B_L is disjoint from $W_F(\ell_1(x_k), \ell_2(x_k))$. This is impossible because B_L contains more than $(\frac{1}{8} - 19\epsilon)n$ points of P_1 that together with the k points of P_1 above m sum up to more than $(\frac{3}{8} - 29\epsilon)n$ points of P_1 non of which lies in $W_F(\ell_1(x_k), \ell_2(x_k))$ which by itself contains at least $(\frac{3}{4} - 10\epsilon)n$ points of P. We thus get a total of $(\frac{9}{8} - 39\epsilon)n > n$ points in P which contradicts the choice of $\epsilon < \frac{1}{8 \cdot 39}$. $\square$

In a symmetric manner one can show the following:

Claim 3. *x_k must be contained in the open interior of $W_F(\ell_1(x_{k+1}), \ell_2(x_{k+1}))$.*

Next we argue symmetrically about P_2. Specifically, we denote the points of P_2 by $y_1, \ldots, y_{\frac{n}{2}}$ according to the increasing order of their x-coordinates. For any point $y \in P_2$, $W_F(\ell_1(y), \ell_2(y))$ is contained either in the half-plane to the left of the horizontal line through y, or in the half-plane to the right of the horizontal line through y. In the former case we say that *y points to the left* and in the latter case we say that *y points to the right.*

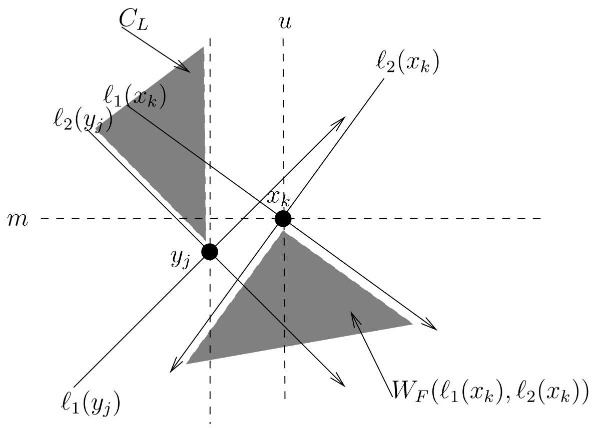

Fig. 5. If $y_j \notin W_F(\ell_1(x_k), \ell_2(x_k))$, then $C_L \cap W_F(\ell_1(x_k), \ell_2(x_k)) = \emptyset$.

Here too, there exists an index $1 \leqslant j < \frac{n}{2}$, such that y_j points to the right but y_{j+1} points to the left. Just like in the case of the index k, one can show that $(\frac{1}{4} - 10\epsilon)n \leqslant j < (\frac{1}{4} + 10\epsilon)n$. Let w be the vertical line through y_j. Denote by C_L the area to the left of w that is contained in $W_L(\ell_1(y_j), \ell_2(y_j))$. Denote by C_R the area to the left of w that is contained in $W_R(\ell_1(y_j), \ell_2(y_j))$. Then one can show that there are at least $(\frac{1}{8} - 19\epsilon)n$ points of P_2 in each of C_L and C_R.

In a similar way one defines w' to be the vertical line through y_{j+1} and let D_L denote the area to the right of w' that is contained in $W_L(\ell_1(y_{j+1}), \ell_2(y_{j+1}))$, and let D_R denote the area to the right of w' that is contained in $W_R(\ell_1(y_{j+1}), \ell_2(y_{j+1}))$. Then there are at least $(\frac{1}{8} - 19\epsilon)n$ points of P_2 in each of D_L and D_R.

Similar to Claims 2 and 3, one can show that y_{j+1} must be contained in the open interior of $W_F(\ell_1(y_j), \ell_2(y_j))$ and that y_j must be contained in the open interior of $W_F(\ell_1(y_{j+1}), \ell_2(y_{j+1}))$.

We need a final key observation to finish the proof of Theorem 2:

Claim 4. *y_j must be contained in the open interior of $W_F(\ell_1(x_k), \ell_2(x_k))$.*

Proof. Let u be the vertical line through x_k. We first show that y_j must lie to the left of u. Indeed, recall that y_j points to the right. This means that if y_j lies to the right of u, then $W_F(\ell_1(y_j), \ell_2(y_j))$ is disjoint from A_R. This is a contradiction because $W_F(\ell_1(y_j), \ell_2(y_j))$ contains more than $(\frac{3}{4} - 10\epsilon)n$ points of P, A_R contains more than $(\frac{1}{8} - 19\epsilon)n$ points of P_1 and to the left of y_j there are at least $j \geqslant (\frac{1}{4} - 10\epsilon)n$ points of P_2. Altogether this sums up to more than $(\frac{9}{8} - 39\epsilon)n \geqslant n$ which is an absurdity. Here we use the assumption that $\epsilon \leqslant \frac{1}{8 \cdot 39}$.

Therefore, if y_j is not contained in the interior of $W_F(\ell_1(x_k), \ell_2(x_k))$, then it must lie to the left of u and necessarily $W_F(\ell_1(x_k), \ell_2(x_k))$ is disjoint from C_L (see Fig. 5). Again we obtain a contradiction for the choice of $\epsilon < \frac{1}{8 \cdot 39}$. $\square$

Similar to Claim 4, one can show that both y_j and y_{j+1} must lie in the open interior of both $W_F(\ell_1(x_k), \ell_2(x_k))$ and $W_F(\ell_1(x_{k+1}), \ell_2(x_{k+1}))$. Also, both x_k and x_{k+1} must lie in the open interior of both $W_F(\ell_1(y_j), \ell_2(y_j))$ and $W_F(\ell_1(y_{j+1}), \ell_2(y_{j+1}))$.

We are now ready to complete the proof of Theorem 2. We claim that the four points x_k, x_{k+1}, y_j, and y_{j+1} must be in convex position. Indeed, this follows because each of these points lies on the boundary of a convex region that contains the three others. Here we mean that x_k lies on the boundary of $W_F(\ell_1(x_k), \ell_2(x_k))$ which contains x_{k+1}, y_j, and y_{j+1} and similarly we argue about each of x_{k+1}, y_j, and y_{j+1}.

Next, consider the convex quadrangle Q whose vertices are x_k, x_{k+1}, y_j, and y_{j+1}. The angle at each of the vertices must be acute. Indeed, consider x_k for example. We know that x_{k+1}, y_j, and y_{j+1} all lie in the *interior* of $W_F(\ell_1(x_k), \ell_2(x_k))$. This implies that the inner angle of Q at x_k must be acute. This is a contradiction because at least one of the four inner angles of any convex quadrangle must be non-acute.

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